HEAT TRANSFER IN THE CASE OF A SEMIINFINITE BODY HEATED BY THIN PARALLEL PLATES

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An exact solution is obtained to the steady-state heat conduction problem with mixed boundary conditions. The solution is carried out by the author's own method shown in earlier publications [4-6].

A solution is sought to the following two-dimensional steady-state heat conduction problem.

On a semiinfinite body are mounted n + 1 thin parallel plates, each of which is held at a constant temperature T_0, T_1, . . . , T_n. These plates are oriented perpendicularly to the flat surface of the body, but the spacing of their fins is arbitrary (Fig. 1a). The heat is transmitted through the flat surface of the body. The ambient temperature is zero.

Mathematically the problem is formulated as follows. Let the sought temperature be T = T(x, y). We then require the solution to the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1}$$

in the upper half-plane with sections at which the following constraints apply:

$$\begin{aligned} T|_{x=a_0} &= T_0, \quad b_0 < y < \infty, \\ T|_{x=a_1} &= T_1, \quad b_1 < y < \infty, \\ &\cdots\cdots\cdots\cdots \\ T|_{x=a_n} &= T_n, \quad b_n < y < \infty. \end{aligned} \tag{2a}$$

Here (a_0, b_0), (a_1, b_1), . . . , (a_n, b_n) are the coordinates of the section endpoints. The following boundary condition of the third kind is satisfied at the interface with the ambient medium:

$$-\frac{\partial T}{\partial y} + hT\Big|_{y=0} = 0, \quad -\infty < x < \infty. \tag{2b}$$

We will seek a solution to the problem which is continuous up to the interface and is bounded at infinity.

The method of solution is as follows. First the original region is mapped conformally onto the first quadrant. This is done with the aid of the Christoffel–Schwarz transformation. At that time the heat omitting boundary becomes the real semiaxis in the new complex plane, while the vertical beams become segments on the imaginary axis (Fig. 1b). Then we construct in this quadrant a function whose real part is the solution to our problem (1)-(2). The procedure for constructing such a function has been described earlier in [4, 5, 7].

This conformal mapping of the original region of the z = x + iy plane onto a quadrant of the w = u + iv plane is achieved by means of the relation

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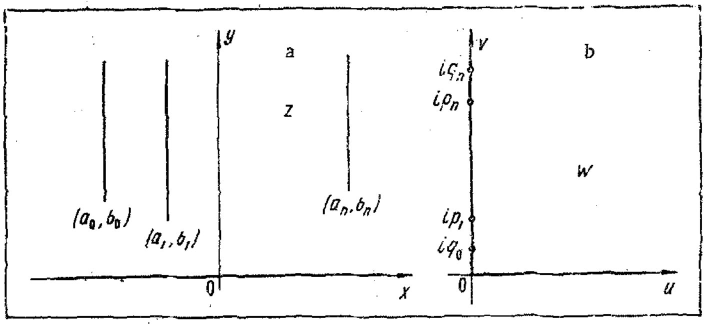

Fig. 1. a) Region for which a solution is sought; b) this region after conformal mapping.

$$z = C\int_1^w \frac{1}{\zeta^2} \frac{\prod_{k=0}^{n}(\zeta^2+q_k^2)}{\prod_{k=1}^{n}(\zeta^2+p_k^2)} d\zeta. \tag{3}$$

Here C is a real positive constant. The section endpoints $z = a_k + ib_k$ $(k = 0, 1, \ldots, n)$ are mapped into points $w = iq_k$ $(k = 0, 1, \ldots, n)$ on the imaginary axis. The section endpoints tending to infinity are mapped into points $w = ip_k$ $(k = 1, 2, \ldots, n)$ on the imaginary axis and, thus, the section edge from point $z = a_0 + ib_0$ to infinity is mapped into segment $u = 0$ $(0 < v < p_1)$ on the imaginary axis, etc. In this way, points $w = iq_k$ $(k = 0, 1, \ldots, n)$ and points $w = ip_k$ $(k = 1, 2, \ldots, n)$ alternate on the imaginary axis. The original region contains two right angles whose vertices are located at infinity. Function

$$z = z(w)$$

has been constructed so that the vertex of one right angle in the original region is mapped into the origin of coordinates in the w-plane, while the vertex of the other right angle is mapped into a point at infinity. Point $z = 0$ is mapped into point $w = 1$ and, therefore, there is no second constant in the transform (3). Points iq_k $(k = 0, 1, \ldots, n)$ and ip_k $(k = 1, 2, \ldots, n)$ and constant C can now be determined uniquely according to the well-known rules of the Christoffel–Schwarz transformation.

The Laplace equation is invariant with respect to the transformation of coordinates involved in conformal mapping. Therefore, function $T = T(x(u, v), y(u, v))$ satisfies the Laplace equation:

$$\frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2} = 0. \tag{4}$$

The boundary conditions for function T are

$$\begin{aligned} T|_{u=0} &= T_0, \quad 0 < v < p_1, \\ T|_{u=0} &= T_1, \quad p_1 < v < p_2, \\ &\cdots\cdots\cdots\cdots \\ T|_{u=0} &= T_n, \quad p_n < v < \infty. \end{aligned} \tag{5a}$$

The boundary condition at the wall is transformed as follows:

$$-\frac{1}{|z'_w|}\cdot\frac{\partial T}{\partial v} + hT\Big|_{v=0} = 0, \quad 0 < u < \infty,$$

which together with (3) yields

$$-u^2 \prod_{k=1}^{n}(u^2+p_k^2)\frac{\partial T}{\partial v} + hC\prod_{k=0}^{n}(u^2+q_k^2)\,T\Big|_{v=0} = 0, \quad 0 < u < \infty. \tag{5b}$$

We will now follow the procedure outlined in [1-3]. We seek a solution to the problem in the form

$$T = \operatorname{Re}\Psi(w), \tag{6}$$

where function $\Psi(w)$ is analytic within any finite $0 < \arg w < \pi/2$ region of the complex $w = u + iv$ plane. According to (5a) and (5b), this function must satisfy the boundary conditions

$$\begin{aligned} &\operatorname{Re}\Psi(w)|_{w=iv} = T_0, \quad 0<v<p_1, \\ &\operatorname{Re}\Psi(w)|_{w=iv} = T_1, \quad p_1<v<p_2, \\ &\dots\dots\dots\dots\dots\dots \\ &\operatorname{Re}\Psi(w)|_{w=iv} = T_n, \quad p_n<v<\infty; \end{aligned} \tag{7a}$$

$$\operatorname{Re}\left[-iw^2\prod_{k=1}^{n}(w^2+p_k^2)\frac{d\Psi(w)}{dw} + hC\prod_{k=0}^{n}(w^2+q_k^2)\Psi(w)\right]\Bigg|_{w=u} = 0, \quad 0<u<\infty. \tag{7b}$$

Function $\Psi(w)$ has a logarithmic singularity at the boundary points $w = 0$, $w = ip_k$ ($k = 1, 2, \ldots, n$), and at infinity. For this reason, it is worthwhile to replace $\Psi(w)$ by a new analytic function $X(w)$ which would satisfy the homogeneous boundary condition on the imaginary axis and which would be continuous within a closed region. Functions $\Psi(w)$ and $X(w)$ are related as follows:

$$\Psi(w) = X(w) + \frac{2}{\pi i}T_0\ln w - \frac{1}{\pi i}\sum_{k=1}^{n}(T_{k-1}-T_k)\left[\ln(w+ip_k)+\ln(w-ip_k)\right]. \tag{8}$$

The boundary conditions for function $X(w)$ become

$$\operatorname{Re}X(w)|_{w=iv} = 0, \quad 0<v<\infty, \tag{9a}$$

$$\operatorname{Re}\left[-iw^2\prod_{k=1}^{n}(w^2+p_k^2)\frac{dX(w)}{dw} + hC\prod_{k=0}^{n}(w^2+q_k^2)X(w)\right]\Bigg|_{w=u} = \frac{2}{\pi}\left\{uT_0 - u^3\sum_{k=1}^{n}\frac{T_{k-1}-T_k}{u^2+p_k^2}\right\}\prod_{k=1}^{n}(u^2+p_k^2), \quad 0<u<\infty. \tag{9b}$$

Function $X(w)$ satisfies the following condition at infinity:

$$X(w) = O(|w|^{-1}), \quad |w|\to\infty, \quad 0\leqslant\arg w\leqslant\pi/2. \tag{10}$$

We note that the coefficients of $dX(w)/dw$ and $X(w)$ in (9b) contain only even powers of w. In this case it is possible then to apply the suggested procedure for finding a function which is analytic in the half-plane and which satisfies at the boundary a condition of the same kind as condition (9b) [4, 5].

The expression inside the square brackets in (9b) represents a function which is analytic within the first quadrant and continuous on the boundary. We will seek this function in the form

$$-iw^2\prod_{k=1}^{n}(w^2+p_k^2)\frac{dX(w)}{dw} + hC\prod_{k=0}^{n}(w^2+q_k^2)X(w) = \frac{2}{\pi}\left[wT_0 - w^3\sum_{k=1}^{n}\frac{T_{k-1}-T_k}{w^2+p_k^2}\right]\prod_{k=1}^{n}(w^2+p_k^2) + i\sum_{k=0}^{n}\alpha_k w^{2k}. \tag{11}$$

Here α_k ($k = 0, 1, \ldots, n$) are certain real constants. They will be determined from the conditions that function $X(w)$, which is defined as the solution to the ordinary differential equation (11), be bounded within the first quadrant and satisfy the boundary condition (9a).

For integrating (11) we use the representation

$$\frac{1}{w^2}\cdot\frac{\prod_{k=0}^{n}(w^2+q_k^2)}{\prod_{k=1}^{n}(w^2+p_k^2)} = 1 + \frac{A}{w^2} + \frac{2}{\pi C}\sum_{k=1}^{n}\frac{(a_k-a_{k-1})\,p_k}{w^2+p_k^2}, \tag{12}$$

where

$$A = \frac{\prod_{k=0}^{n} q_k^2}{\prod_{k=1}^{n} p_k^2}. \tag{13}$$

After that, we obtain

$$X(w) = \exp\left(-ihC\left(w - \frac{A}{w}\right)\right) \prod_{k=1}^{n} \left(\frac{w + ip_k}{w - ip_k}\right)^{(\alpha_k - \alpha_{k-1})h/\pi}$$
$$\times \int_{w}^{\infty} \exp\left(ihC\left(\zeta - \frac{A}{\zeta}\right)\right)\left[\prod_{k=1}^{n}\left(\frac{\zeta - ip_k}{\zeta + ip_k}\right)^{(\alpha_k - \alpha_{k-1})h/\pi}\right] F(\zeta)\, d\zeta, \tag{14}$$

with (10) taken into account and with the following designation:

$$F(\zeta) = \frac{2}{\pi i}\left[\frac{T_0}{\zeta} - \zeta \sum_{k=1}^{n} \frac{T_{k-1} - T_k}{\zeta^2 + p_k^2}\right] + \frac{1}{\zeta^2} \cdot \frac{\sum_{k=0}^{n} \alpha_k \cdot \zeta^{2k}}{\prod_{k=1}^{n} (\zeta^2 + p_k^2)}. \tag{15}$$

In (14) we choose those branches of the function $(w - ip_k)^{(\alpha_k - \alpha_{k-1})h/\pi}$, $(w + ip_k)^{(\alpha_k - \alpha_{k-1})h/\pi}$ ($k = 1, 2, \ldots, n$) which at $w \to \infty$ tend toward $u^{(\alpha_k - \alpha_{k-1})h/\pi}$ along the positive real axis. The integration is performed along any path in the first quadrant not passing through point $w = 0$.

Function $X(w)$ (14) with an arbitrary set of constants α_k ($k = 0, 1, \ldots, n$) is continuous at all points on the boundary, except at points $w = 0$ and $w = ip_k$ ($k = 1, 2, \ldots, n$). In satisfying the condition

$$\int_{ip_k}^{\infty} \exp\left(ihC\left(\zeta - \frac{A}{\zeta}\right)\right)\left[\prod_{s=1}^{n}\left(\frac{\zeta - ip_s}{\zeta + ip_s}\right)^{(\alpha_s - \alpha_{s-1})h/\pi}\right] F(\zeta)\, d\zeta = 0 \quad (k = 1, 2, \ldots, n) \tag{16}$$

points $w = ip_k$ are removable singularities of function $X(w)$. When considering the behavior of function $X(w)$ (14) in the vicinity of point $w = 0$, we note that this point is an essential singularity of the function

$$\exp\left(-ihC\left(w - \frac{A}{w}\right)\right).$$

We extract sector $0 < \delta \leq \arg w \leq \pi/2$ and we require that

$$\int_{0}^{\infty} \exp\left(ihC\left(\zeta - \frac{A}{\zeta}\right)\right)\left[\prod_{k=1}^{n}\left(\frac{\zeta - ip_k}{\zeta + ip_k}\right)^{(\alpha_k - \alpha_{k-1})h/\pi}\right] F(\zeta)\, d\zeta = 0, \tag{17}$$

where the integration is performed along any curve within this sector. When condition (17) is satisfied, then the following limiting process is valid:

$$\lim_{w \to 0} X(w) = \frac{i\alpha_0}{hC \prod_{k=0}^{n} q_k^2}, \quad 0 < \delta \leqslant \arg w \leqslant \frac{\pi}{2}. \tag{18}$$

The integration in (16) and (17) can be performed along the imaginary axis. Successively subtracting the integrals, we arrive at the following system of equations:

$$\int_{ip_k}^{ip_{k+1}} \exp\left(ihC\left(\zeta - \frac{A}{\zeta}\right)\right)\left[\prod_{s=1}^{n}\left(\frac{\zeta - ip_s}{\zeta + ip_s}\right)^{(\alpha_s - \alpha_{s-1})h/\pi}\right] F(\zeta) \cdot d\zeta = 0 \quad (k = 0, 1, \ldots, n) \tag{19}$$

where $p_0 = 0$ and where ip_{n+1} denotes a point removed to infinity. Changing to a real variable of integration in (19) will yield a system of n + 1 linear algebraic equations with real coefficients for determining n constants α_k (k = 0, 1, . . . , n).

With this choice of constants, the limiting process (18) is valid also for $w \to 0$ along the positive real axis.

When condition (16) is satisfied, ensuring the continuity of function X(w) (14) at points $w = ip_k$ (k = 1, 2, . . . , n) on the boundary, the boundary condition (9a) will also be satisfied on the entire imaginary axis. This can be proved easily, if ReX(w) is extracted in each segment $u = 0$, $p_k < v < p_{k+1}$ (k = 0, 1, . . . , n).

An asymptotic estimate of (10) is made on the basis of formula (14), by integrating in parts.

Thus, function X(w) (14) is regular within the first quadrant and, by virtue of its properties, function T = T(x, y) defined by (6), (8), (14) satisfies all conditions (1)-(2) stipulated in the problem.

For illustration, we will consider the solution to a specific problem where the heat source is a single plate held at the temperature T_0. In this case conditions (2) become

$$T|_{x=0} = T_0, \quad b < y < \infty, \tag{20a}$$

$$-\frac{\partial T}{\partial y} + hT\Big|_{y=0} = 0, \quad -\infty < x < \infty, \tag{20b}$$

and the mapping function (3) is

$$z = \frac{b}{2}\left(w - \frac{1}{w}\right). \tag{21}$$

After a change to new variables, the problem reduces to one of determining a function $\Psi(w)$ which is regular in the $0 < \arg w < \pi/2$ quadrant and which satisfies the conditions

$$\operatorname{Re}\Psi(w)|_{w=iv} = T_0, \quad 0 < v < \infty, \tag{22a}$$

$$\operatorname{Re}\left[-i2w^2\frac{d\Psi(w)}{dw} + bh(w^2+1)\Psi(w)\right]\Big|_{w=u} = 0, \quad 0 < u < \infty. \tag{22b}$$

Assuming, as before, that

$$\Psi(w) = X(w) + \frac{2}{\pi i} T_0 \ln w \tag{23}$$

and applying the conventional method, we obtain

$$X(w) = T_0 \exp\left(-i\frac{bh}{2}\left(w - \frac{1}{w}\right)\right)\int_w^\infty \exp\left(i\frac{bh}{2}\left(\zeta - \frac{1}{\zeta}\right)\right)\left[\frac{\alpha}{2\zeta^2} + \frac{2}{\pi i}\cdot\frac{1}{\zeta}\right]d\zeta. \tag{24}$$

The constant α here is determined from the condition that X(w) is finite at point w = 0, which results in the equation

$$\int_0^\infty \exp\left(i\frac{bh}{2}\left(\zeta - \frac{1}{\zeta}\right)\right)\left[\frac{\alpha}{2\zeta^2} + \frac{2}{\pi i}\cdot\frac{1}{\zeta}\right]d\zeta = 0. \tag{25}$$

From this follows

$$\alpha = -\frac{4}{\pi}\cdot\frac{K_0(bh)}{K_1(bh)}, \tag{26}$$

where $K_0(bh)$, $K_1(bh)$ are MacDonald functions.

For the temperature distribution over the body surface we have obtained the following expression:

$$T(x, 0) = T_0\frac{2}{\pi}\left\{\int_x^\infty \frac{\sin h(t-x)}{\sqrt{t^2+b^2}}dt - \frac{1}{b}\cdot\frac{K_0(bh)}{K_1(bh)}\int_x^\infty \cos h(t-x)\left[1 - \frac{t}{\sqrt{t^2+b^2}}\right]dt\right\} \tag{27}$$

$$(0 \leqslant x < \infty).$$

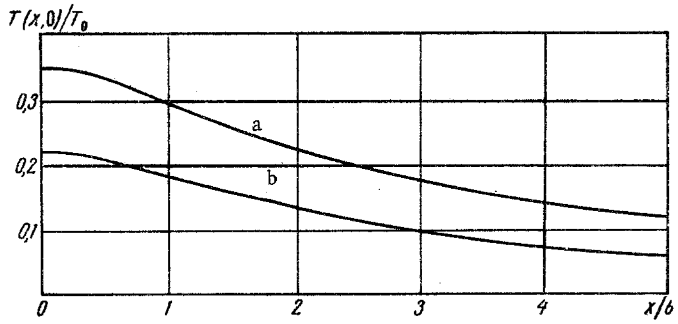

Fig. 2. Temperature distribution along the heat emitting wall: a) hb = 1; b) hb = 2.

Specifically, for large values of x we have the asymptotic formula

$$T(x,\ 0) \approx T_0 \frac{2}{\pi} \cdot \frac{1}{hx},\ x \to \infty. \tag{28}$$

Numerical computations have been performed for the following parameter combinations: a) hb = 1; b) hb = 2. The results are shown in Fig. 2.

In conclusion, we note that for this special case n = 0 (a single plate) the problem can be solved directly by a separation of variables in elliptical coordinates. A determination of the expansion coefficient in the integral representation of the sought function reduces to the solution of certain functional-difference equation. The results obtained that way agree with the results of our calculation here. The method of separating variables cannot, however, be extended to the general case of any number of heating plates.

NOTATION

T_k (k = 0, 1, . . . , n)	are the temperatures of the parallel plates;
T	is the temperature within the region;
h	is the heat-transfer coefficient;
a_k, b_k (k = 0, 1, . . . , n)	is the firing constant at the plates.

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